

Generic extensions of nilpotent $k[T]$ -modules, monoids of partitions and constant terms of Hall polynomials

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Abstract. We prove that the monoid of generic extensions of finite dimensional nilpotent $k[T]$ -modules is isomorphic to the monoid of partitions (with addition of partitions). Moreover we give a combinatorial algorithm that calculates constant terms of classical Hall polynomials.

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1. Introduction

Let k be an algebraically closed field and let $k[T]$ be the k -algebra of polynomials in one variable T . We consider nilpotent $k[T]$ -modules M , N and the generic extension $M * N$ of M by N , i.e. an extension of M by N with the minimal dimension of its endomorphism ring (see Section 2 for definitions). By results presented in [3, 4, 11] generic extensions of nilpotent $k[T]$ -modules exist and the operation of taking the generic extension provides the set of all isomorphism classes of nilpotent $k[T]$ -modules with a monoid structure \mathcal{M}^* . In this paper we study connections of this monoid with the monoid \mathcal{P}^+ of all partitions with addition of partitions as an action. More precisely, we prove in Theorem 3.1 that these monoids are isomorphic. This isomorphism gives us a combinatorial description of generic extensions that have a geometric nature. For a geometric interpretation of generic extensions the reader is referred to [11, 12].

On the other hand there is a \mathbb{C} -algebra isomorphism $\mathbb{C}\mathcal{M}^* \simeq \mathcal{H}_0$, where \mathcal{H}_0 is the specialisation of the Hall algebra \mathcal{H}_q to $q = 0$ and $\mathbb{C}\mathcal{M}^*$ is the \mathbb{C} -algebra generated by the monoid \mathcal{M}^* (see [4, 5, 13] and Section 3). There are many results that show connections between generic extensions, Hall polynomials and Ringel-Hall algebras (see [11, 12, 5, 13] for Dynkin and extended Dynkin quivers, [4] for cyclic quivers, [6, 7] for poset representations).

In Section 4, exploring the isomorphism $\mathbb{C}\mathcal{M}^* \simeq \mathcal{H}_0$ (explicitly given in [13]), we describe a combinatorial algorithm that finds the constant terms of classical

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2. Notation and definitions

Throughout this paper k is a fixed algebraically closed field.

Let $\lambda = (\lambda_1, \dots, \lambda_n, \dots)$ be a partition (i.e. a sequence of non-negative integers containing only finitely many non-zero terms and such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq \dots$). Denote by $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_n, \dots)$ the dual partition of λ , i.e.

$$\bar{\lambda}_i = \#\{j ; \lambda_j \geq i\},$$

where $\#X$ denotes the cardinality of a finite set X . We identify partitions that differ only by a string of zeros at the end. Let \mathcal{P} be the set of all partitions. Denote by $|\lambda|$ the **weight** of λ defined by

$$|\lambda| = \lambda_1 + \lambda_2 + \dots$$

and by $0 = (0)$ the unique partition of zero. Consider two associative monoids:

- $\mathcal{P}^+ = (\mathcal{P}, +, 0)$, where $(\lambda_1, \lambda_2, \dots) + (\nu_1, \nu_2, \dots) = (\lambda_1 + \nu_1, \lambda_2 + \nu_2, \dots)$;
- $\mathcal{P}^\cup = (\mathcal{P}, \cup, 0)$, where $(\lambda_1, \lambda_2, \dots) \cup (\nu_1, \nu_2, \dots) = (\mu_1, \mu_2, \dots)$ and (μ_1, μ_2, \dots) is the partition that is consisted of integers $\lambda_1, \lambda_2, \dots, \nu_1, \nu_2, \dots$ arranged in the descending order (e.g. $(3, 3, 2, 1) + (2, 2) = (5, 5, 2, 1)$ and $(3, 3, 2, 1) \cup (2, 2) = (3, 3, 2, 2, 2, 1)$);

By [10, 1.8] the operations $+$ and \cup are dual to each other (i.e. $\overline{\lambda \cup \nu} = \bar{\lambda} + \bar{\nu}$). One of the main aims of the paper is to describe connections of these monoids with extensions of nilpotent $k[T]$ -modules.

Let $k[T]$ be the k -algebra of polynomials in the variable T . For any partition $\lambda = (\lambda_1, \dots, \lambda_n, \dots)$, where $\lambda_{n+1} = \lambda_{n+2} = \dots = 0$, denote by

$$M(\lambda) = M(\lambda, k) \cong k[T]/(T^{\lambda_1}) \oplus \dots \oplus k[T]/(T^{\lambda_n})$$

the corresponding $k[T]$ -module. It is obvious that the function $\lambda \rightarrow M(\lambda)$ gives a bijection between the set \mathcal{P} of all partitions and the set of all isomorphism classes of nilpotent $k[T]$ -modules (i.e. finitely generated $k[T]$ -modules M such that $T^a M = 0$ for some $a \geq 0$). Denote by \mathcal{M} a set of representatives of all isomorphism classes of nilpotent $k[T]$ -modules.

Let $M, N \in \mathcal{M}$. By [3], [4] and [11], there is the unique (up to isomorphism) extension X of M by N with the minimal dimension of endomorphism ring $\text{End}_{k[T]}(X)$, i.e. a nilpotent $k[T]$ -module X such that there exists a short exact sequence of the form

$$0 \rightarrow N \rightarrow X \rightarrow M \rightarrow 0.$$

The module X is called the **generic extension** of M by N and is denoted by $X = M * N$. Denote by $M \oplus N$ the direct sum of the modules M and N and by 0 the unique zero module. Consider two monoids:

- $\mathcal{M}^* = (\mathcal{M}, *, 0)$ (**the monoid of generic extensions**),
- $\mathcal{M}^\oplus = (\mathcal{M}, \oplus, 0)$.

The associativity of the monoid $\mathcal{M}^* = (\mathcal{M}, *, 0)$ follows by [4], whereas that of the monoid $\mathcal{M}^\oplus = (\mathcal{M}, \oplus, 0)$ is obvious.

3. Generic extensions and partitions

The following fact is one of the main results of the paper.

THEOREM 3.1 *The function*

$$\Phi : \mathcal{P} \rightarrow \mathcal{M}$$

such that $\Phi(\lambda) = M(\lambda)$, for any partition λ , induces isomorphisms of monoids:

$$\Phi : \mathcal{P}^+ \rightarrow \mathcal{M}^*$$

and

$$\Phi : \mathcal{P}^\cup \rightarrow \mathcal{M}^\oplus.$$

Moreover

$$M(\overline{\alpha}) * M(\overline{\beta}) = M(\overline{\alpha \cup \beta}).$$

To prove Theorem 3.1 we need a geometric interpretation of generic extensions.

We identify $k[T]$ -modules of the form $M(\lambda, k)$ with systems $M(\lambda, k) = (V, \varphi)$, where V is a finite dimensional k -vector space and $\varphi : V \rightarrow V$ is a nilpotent linear endomorphism with the Jordan type λ (i.e. nilpotent representation of a loop quiver). By $\mathcal{N}(k)$ we denote the category of all such systems. If (V, φ) , (V_1, φ_1) are objects in $\mathcal{N}(k)$, then a morphism $f : (V, \varphi) \rightarrow (V_1, \varphi_1)$ is a linear map $f : V \rightarrow V_1$ such that $\varphi_1 f = f \varphi$. It is easy to see that the category $\mathcal{N}(k)$ is equivalent to the category of all finite dimensional nilpotent $k[T]$ -modules. For an account of the theory of modules and quiver representations we refer the reader to [1] and [2].

Consider the affine variety $\mathbb{M}_n(k)$ of all $n \times n$ -matrices with coefficients in k . The general linear group $\mathrm{GL}_n(k)$ acts on $\mathbb{M}_n(k)$ via conjugations, i.e. for $g \in \mathrm{GL}_n(k)$ and $M \in \mathbb{M}_n(k)$, we put $g \cdot M = g M g^{-1}$. Let $\mathbb{M}_n^{nil}(k)$ be the subset of $\mathbb{M}_n(k)$ consisted of all nilpotent matrices. The subset $\mathbb{M}_n^{nil}(k)$ is closed in $\mathbb{M}_n(k)$ (in Zariski topology) and it is closed under the action of $\mathrm{GL}_n(k)$. It is easy to observe that points of $\mathbb{M}_n^{nil}(k)$ corresponds bijectively to the objects (V, φ) of $\mathcal{N}(k)$ with $\dim_k V = n$. Moreover the orbits of the action of $\mathrm{GL}_n(k)$ on $\mathbb{M}_n^{nil}(k)$ corresponds bijectively to the isomorphism classes of the objects V in $\mathcal{N}(k)$ (with $\dim_k V = n$) and hence to the isomorphism classes of finite dimensional nilpotent $k[T]$ -modules V (with $\dim_k V = n$). If $M(\lambda) \equiv (V, \varphi)$ is a nilpotent $k[T]$ -module with $\dim_k M(\lambda) = n$, then we denote by \mathcal{O}_λ (resp. $\overline{\mathcal{O}_\lambda}$) the orbit (resp. the Zariski-closure) of $\varphi \in \mathbb{M}_n^{nil}(k)$ of the $\mathrm{GL}_n(k)$ -action.

Let λ, ν be partitions with weights $|\lambda| = |\nu| = n$. We say that a module $M(\lambda)$ **degenerates** to the module $M(\nu)$, if $\mathcal{O}_\nu \in \overline{\mathcal{O}_\lambda}$. If $M(\lambda)$ degenerates to $M(\nu)$ we write $M(\lambda) \leq_{deg} M(\nu)$. The relation \leq_{deg} is a partial order on isomorphism classes of finite dimensional nilpotent $k[T]$ -modules. Geometrically, the generic extension $M(\lambda) * M(\nu)$ (resp. the direct sum $M(\lambda) \oplus M(\nu)$) is the \leq_{deg} -minimal (resp. \leq_{deg} -maximal) extension of $M(\nu)$ by $M(\lambda)$, i.e. if X is an extension of $M(\nu)$ by $M(\lambda)$, then $M(\lambda) * M(\nu) \leq_{deg} X$ (resp. $X \leq_{deg} M(\lambda) \oplus M(\nu)$), see [3], [4] and [11]. For an introduction to geometric methods in representation theory the reader is referred to [8] and [3].

The following fact is proved in [9, I.3].

THEOREM 3.2 *Let λ, ν be partitions with $|\lambda| = |\nu|$. $M(\lambda) \leq_{deg} M(\nu)$ if and only if, for any $m \geq 1$:*

$$\sum_{i=1}^m \bar{\lambda}_i \leq \sum_{i=1}^m \bar{\nu}_i.$$

The following lemma is used in the proof of Theorem 3.1.

LEMMA 3.3 *Let σ, ν, μ be partitions. If there exists a short exact sequence*

$$0 \longrightarrow M(\nu) \xrightarrow{a} M(\sigma) \xrightarrow{b} M(\mu) \longrightarrow 0,$$

then for any $m \geq 1$:

$$\sum_{i=1}^m \sigma_i \leq \sum_{i=1}^m \lambda_i,$$

where $\lambda = \mu + \nu$.

Proof. The proof is by induction on $|\nu|$. If $|\nu| = 0$, then $M(\sigma) \cong M(\mu)$, $\sigma = \mu$ and we are done.

Assume that $|\nu| > 0$. We have $\nu = (\nu_1, \dots, \nu_n)$, $\nu_n \neq 0$ and

$$M(\nu) = M(\nu_1) \oplus \dots \oplus M(\nu_n).$$

Consider the monomorphism

$$f = [0, 0, \dots, 0, \iota] : M(1) \rightarrow M(\nu_1) \oplus \dots \oplus M(\nu_n),$$

where $\iota : M(1) \rightarrow M(\nu_n)$ is an inclusion. By the Snake Lemma, we get the following diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & M(\nu') & \longrightarrow & M(\sigma') & \longrightarrow & M(\mu) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & M(\nu) & \xrightarrow{a} & M(\sigma) & \xrightarrow{b} & M(\mu) & \longrightarrow & 0 \\
 & & \uparrow f & & \uparrow a \cdot f & & \uparrow 0 & & \\
 0 & \longrightarrow & M(1) & \xrightarrow{\text{id}} & M(1) & \xrightarrow{0} & 0 & \longrightarrow & 0 \\
 & & \uparrow 0 & & \uparrow 0 & & & & \\
 & & 0 & & 0 & & & &
 \end{array}$$

where $\nu' = (\nu_1, \nu_2, \dots, \nu_{n-1}, \nu_n - 1)$ and there exists i such that $\sigma'_i = \sigma_i - 1$ and $\sigma'_j = \sigma_j$ for $j \neq i$. By the induction hypothesis we get

$$\sum_{i=1}^m \sigma'_i \leq \sum_{i=1}^m \lambda'_i,$$

where $\lambda' = \mu + \nu'$. Therefore

$$\sum_{i=1}^m \sigma_i \leq \sum_{i=1}^m \lambda_i,$$

where $\lambda = \mu + \nu$ and we are done. \square

LEMMA 3.4 *Let ν, μ be partitions. We have*

$$M(\nu) * M(\mu) = M(\nu + \mu),$$

where $M(\nu) * M(\mu)$ is the generic extension of $M(\nu)$ by $M(\mu)$.

Proof. It is easy to see that $M(\nu + \mu)$ is an extension of $M(\nu)$ by $M(\mu)$. If $M(\sigma)$ is any extension of $M(\mu)$ by $M(\nu)$, then by Lemma 3.3, for any $m \geq 1$:

$$\sum_{i=1}^m \sigma_i \leq \sum_{i=1}^m \lambda_i,$$

where $\lambda = \mu + \nu$. By [10, 1.11], for any $m \geq 1$:

$$\sum_{i=1}^m \bar{\sigma}_i \geq \sum_{i=1}^m \bar{\lambda}_i.$$

Theorem 3.2 yields

$$M(\nu + \mu) = M(\lambda) \leq_{deg} M(\sigma).$$

Since $M(\lambda) * M(\nu)$ is the \leq_{deg} -minimal extension of $M(\nu)$ by $M(\mu)$, we are done. \square

Proof of Theorem 3.1. Let

$$\Phi : \mathcal{P} \rightarrow \mathcal{M}$$

be such that $\Phi(\lambda) = M(\lambda)$, for any partition λ . By Lemma 3.4, the induced function

$$\Phi : \mathcal{P}^+ \rightarrow \mathcal{M}^*$$

is an isomorphism of monoids. It is easy to see that

$$\Phi : \mathcal{P}^\cup \rightarrow \mathcal{M}^\oplus.$$

is an isomorphism of monoids. Moreover

$$M(\overline{\alpha}) * M(\overline{\beta}) = M(\overline{\alpha + \beta}) = M(\overline{\alpha \cup \beta}),$$

because $\overline{\alpha} + \overline{\beta} = \overline{\alpha \cup \beta}$. \square

4. Constant terms of Hall polynomials

In this section we describe a combinatorial algorithm that finds the constant term of a given Hall polynomial.

Let α, β, γ be partitions and let k be a finite field. Denote by

$$F_{\alpha, \beta}^\gamma(k)$$

the number of submodules U of $M(\gamma, k)$ that are isomorphic to $M(\beta, k)$ and the factor module $M(\gamma, k)/U$ is isomorphic to $M(\alpha, k)$. By the result of Hall (see [10, II.4.3]), there exists a polynomial $\varphi_{\alpha, \beta}^\gamma$ with integral coefficients such that:

$$\varphi_{\alpha, \beta}^\gamma(\#k) = F_{\alpha, \beta}^\gamma(k)$$

for any finite field k . We call $\varphi_{\alpha, \beta}^\gamma$ the **Hall polynomial** associated with partitions α, β, γ .

By [4], [5] and [13], the complex algebra $\mathbb{C}\mathcal{M}^*$ generated by the monoid \mathcal{M}^* of generic extension is isomorphic to the degenerate complex Hall algebra \mathcal{H}_0 , where \mathcal{H}_0 has a basis

$$\{u_\alpha ; \alpha \in \mathcal{P}\}$$

as a \mathbb{C} -vector space and the multiplication is given by the formula

$$u_\alpha u_\beta = \sum_{\gamma} \varphi_{\alpha, \beta}^\gamma(0) u_\gamma.$$

By [13], the isomorphism

$$F : \mathbb{C}\mathcal{M}^* \rightarrow \mathcal{H}_0$$

is given by the formula

$$F(M(\alpha)) = \sum_{\beta: M(\alpha) \leq_{deg} M(\beta)} u_\beta.$$

We use the following notation. A partition $\alpha = (\alpha_1, \dots, \alpha_n, \dots)$ shall be written as

$$(\dots, r^{m_r}, \dots, 2^{m_2}, 1^{m_1}),$$

where m_r indicates the number of times the integer r occurs in α , e.g.

$$(3, 3, 2, 2, 2, 1, 1, 1, 1) = (3^2, 2^3, 1^4).$$

LEMMA 4.1 *Let γ be an arbitrary partition and let $\alpha = (1^n), \beta = (1^m)$ be partitions with the property $\varphi_{\alpha, \beta}^\gamma \neq 0$. Then*

$$\varphi_{\alpha, \beta}^\gamma(0) = 1.$$

Proof. Note that $F(M(\alpha)) = u_\alpha$ and $F(M(\beta)) = u_\beta$. Then

$$F(M(\alpha))F(M(\beta)) = u_\alpha u_\beta = \sum_{\delta} \varphi_{\alpha, \beta}^\delta(0) u_\delta$$

and

$$F(M(\alpha))F(M(\beta)) = F(M(\alpha) * M(\beta)) = F(M(\alpha + \beta)) = \sum_{M(\alpha + \beta) \leq_{deg} M(\delta)} u_\delta$$

Comparing these sums we get $\varphi_{\alpha, \beta}^\gamma(0) = 1$, if $\varphi_{\alpha, \beta}^\gamma \neq 0$. \square

Applying recursively (following \leq_{deg} -order) methods applied in the proof of Lemma 4.1 one can calculate constant terms of Hall polynomials. We illustrate this algorithm in the following example.

EXAMPLE 4.2 We calculate the constant term of the Hall polynomial $\varphi_{(2,1)(2)}^{(4,1)}$. We apply Theorem 3.2 and the definition of F .

Step 1. By Lemma 4.1, we have

$$\varphi_{(1^3)(1^2)}^{(1^5)}(0) = \varphi_{(1^3)(1^2)}^{(2,1^3)}(0) = \varphi_{(1^3)(1^2)}^{(2^2,1)}(0) = 1.$$

Step 2. Note that

$$\begin{aligned} F(M(1^3))F(M(2)) &= u_{(1^3)}(u_{(1^2)} + u_{(2)}) = \\ &= u_{(1^5)} + u_{(2,1^3)} + u_{(2^2,1)} + \varphi_{(1^3)(2)}^{(2,1^3)}(0)u_{(2,1^3)} + \varphi_{(1^3)(2)}^{(3,1^2)}(0)u_{(3,1^2)}. \end{aligned}$$

On the other hand

$$F(M(1^3) * M((2))) = F(M(3, 1^2)) = u_{(1^5)} + u_{(2,1^3)} + u_{(2^2,1)} + u_{(3,1^2)}.$$

Therefore

$$\varphi_{(1^3)(2)}^{(3,1^2)}(0) = 1 \quad \text{and} \quad \varphi_{(1^3)(2)}^{(2,1^3)}(0) = 0.$$

Step 3. We have

$$\begin{aligned} F(M(2,1))F(M(1^2)) &= (u_{(2,1)} + u_{(1^3)})u_{(1^2)} = \\ &= \varphi_{(2,1)(1^2)}^{(2,1^3)}(0)u_{(2,1^3)} + \varphi_{(2,1)(1^2)}^{(3,1^2)}(0)u_{(3,1^2)} + \varphi_{(2,1)(1^2)}^{(2^2,1)}(0)u_{(2^2,1)} + \varphi_{(2,1)(1^2)}^{(3,2)}(0)u_{(3,2)} + \\ &\quad + u_{(1^5)} + u_{(2,1^3)} + u_{(2^2,1)}. \end{aligned}$$

and

$$F(M(2,1) * M((1^2))) = F(M(3,2)) = u_{(1^5)} + u_{(2,1^3)} + u_{(2^2,1)} + u_{(3,1^2)} + u_{(3,2)}.$$

Therefore

$$\begin{aligned} \varphi_{(2,1)(1^2)}^{(2,1^3)}(0) &= \varphi_{(2,1)(1^2)}^{(2^2,1)}(0) = 0 \\ \varphi_{(2,1)(1^2)}^{(3,1^2)}(0) &= \varphi_{(2,1)(1^2)}^{(3,2)}(0) = 1. \end{aligned}$$

Step 4. Finally

$$\begin{aligned} F(M(2,1))F(M(2)) &= (u_{(2,1)} + u_{(1^3)})(u_{(1^2)} + u_{(2)}) = \\ &= \varphi_{(2,1)(2)}^{(2^2,1)}(0)u_{(2^2,1)} + \varphi_{(2,1)(2)}^{(3,2)}(0)u_{(3,2)} + \varphi_{(2,1)(2)}^{(4,1)}(0)u_{(4,1)} + \varphi_{(2,1)(2)}^{(3,1^2)}(0)u_{(3,1^2)} + \\ &\quad + u_{(3,1^2)} + u_{(3,2)} + u_{(3,1^2)} + u_{(1^5)} + u_{(2,1^3)} + u_{(2^2,1)}. \end{aligned}$$

and

$$F(M(2,1) * M((2))) = F(M(4,1)) = u_{(1^5)} + u_{(2,1^3)} + u_{(2^2,1)} + u_{(3,1^2)} + u_{(3,2)} + u_{(4,1)}.$$

Therefore

$$\begin{aligned} \varphi_{(2,1)(2)}^{(2^2,1)}(0) &= \varphi_{(2,1)(2)}^{(3,2)}(0) = 0 \\ \varphi_{(2,1)(2)}^{(4,1)}(0) &= 1 \\ \varphi_{(2,1)(2)}^{(3,1^2)}(0) &= -1. \end{aligned}$$

REMARK 4.3 In a similar way (exploring isomorphism analogous to F given in [13]) one may calculate constant terms of Hall polynomials for Dynkin quivers.

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